

# Vertex representations of toroidal special linear Lie superalgebras

Naihuan Jing and Chongbin Xu\*

**ABSTRACT.** Based on the loop-algebraic presentation of 2-toroidal Lie superalgebras, free field representation of toroidal Lie superalgebras of type  $A(m, n)$  is constructed using both vertex operators and bosonic fields.

## 1. Introduction

Let  $\mathfrak{g}$  be a finite-dimensional simple Lie (super)algebra of type  $X$  and  $R$  be the ring of Laurent polynomials in commuting variables, the toroidal Lie (super)algebra  $T(\mathfrak{g})$  is by definition the perfect central extension of the loop algebra  $\mathfrak{g} \otimes R$ . When  $R = \mathbb{C}[t, t^{-1}]$ , the toroidal Lie algebra is the affine Kac-Moody Lie algebra. The larger class of Lie (super)algebras  $T(\mathfrak{g})$  shares many properties with the untwisted affine Lie (super)algebras.

In the case of untwisted toroidal Lie algebras, Moody, Rao and Yokonuma [18] gave a loop algebra presentation for the 2-toroidal Lie algebras similar to the affine Kac-Moody Lie algebras, which has set the stage for later developments such as free field realizations and vertex operator representations. Notably in [20] the type  $B_n$  toroidal Lie algebras were constructed using fermionic operators (also cf. [8]). On the other hand level one representations of toroidal Lie algebras of simply laced types were realized via McKay correspondence and wreath products of Kleinian subgroups of  $SL_2(\mathbb{C})$  [5]. Bosonic realizations of higher level toroidal Lie algebras  $T(A_1)$

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\*Corresponding author.

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were also given in [10]. More recently a unified realization [9, 11] of all 2-toroidal Lie algebras of classical types was constructed using bosonic or fermionic fields, which has generalized the Feingold-Frenkel construction [4] for affine Lie algebras.

Affine Lie superalgebras have been studied as early as their non-super counterparts. In fact, Feingold and Frenkel construction works for Lie superalgebras as well [4]. Vertex superalgebras and their representations were also given in [15]. Later in [14] integrable highest weight modules were constructed for affine superalgebras of orthosymplectic series using fermionic and bosonic fields. All these constructions were based on the loop algebra realizations of affine Lie superalgebras.

Irreducible highest weight modules of classical toroidal Lie superalgebras can be constructed abstractly as in the affine cases [19]. Various other constructions of toroidal Lie superalgebras and their generalizations were known [2, 17, 9, 16, 3]. In particular, [1] has constructed certain vertex operator representation for the general toroidal cases. Recently we have given a loop algebra realization for 2-toroidal classical superalgebras [12], which is a super analog of the MRY construction (see [7] for earlier development).

The aim of this work is to generalize Kac and Wakimoto's work on affine superalgebras of unitary series to 2-toroidal setting using both vertex operators and Weyl bosonic fields, and the construction has utilized our recent MRY presentation exclusively. We remark that our work is different from [1] in that we use more bosonic fields while the latter used more vertex operators. This suggests that there could be a super boson-fermion correspondence for the 2-toroidal cases.

The paper is organized as follows. In section 2 we recall the notion of 2-toroidal Lie superalgebras and the loop-algebra presentation. In section 3 we construct certain vertex operators and Weyl bosonic fields to give a level one representation of the 2-toroidal Lie special linear superalgebra.

## 2. Toroidal Lie superalgebra $\mathfrak{T}(A(m, n))$

Let  $V = \mathbb{C}^{m|n+1}$  be the  $\mathbb{Z}_2$ -graded vector space of dimension  $(m, n+1)$ , where  $m \neq n$ . Let  $\mathfrak{gl}(m|n+1)$  be the Lie superalgebra of the super-endomorphisms of  $V$  under the superbracket. Let  $\mathfrak{g}$  be the traceless subalgebra, i.e. the simple Lie superalgebra of type  $A(m, n)$ . Let  $R = \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$  be the complex commutative ring of Laurent polynomials in  $s, t$ . The

loop Lie superalgebra  $L(\mathfrak{g}) := \mathfrak{g} \otimes R$  is defined under the Lie superbracket  $[x \otimes a, y \otimes b] = [x, y] \otimes ab$ .

Let  $\Omega_R$  be the  $R$ -module of Kähler differentials  $\{bda | a, b \in R\}$ , and let  $d\Omega_R$  be the space of exact forms. The quotient space  $\Omega_R/d\Omega_R$  has a basis consisting of  $\overline{s^{m-1}t^n ds}$ ,  $\overline{s^n t^{-1} dt}$ ,  $\overline{s^{-1} ds}$ , where  $m, n \in \mathbb{Z}$ . Here  $\overline{a}$  denotes the coset  $a + d\Omega_R$ .

The toroidal special linear superalgebra  $T(\mathfrak{g})$  is defined to be the Lie superalgebra on the following vector space:

$$T(\mathfrak{g}) = \mathfrak{g} \otimes R \oplus \Omega_R/d\Omega_R$$

with the Lie superbracket  $(x, y \in \mathfrak{g}, a, b \in R)$ :

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab + (x|y)\overline{(da)b}, \quad [T(\mathfrak{g}), \Omega_R/d\Omega_R] = 0$$

and the parities are specified by:

$$p(x \otimes a) = p(x), \quad p(\Omega_R/d\Omega_R) = \overline{0}.$$

Let  $A = (a_{ij})$  be the extended distinguished Cartan matrix of the affine Lie superalgebra of type  $A(m, n)^{(1)}$ , i.e.

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & \cdots & 0 & 1 \\ -1 & 2 & -1 & \ddots & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\ 0 & \cdots & -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & & & 2 & -1 & 0 \\ 0 & \cdots & 0 & & & -1 & 2 & -1 \\ -1 & \cdots & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

and  $Q = \mathbb{Z}\alpha_0 \oplus \cdots \oplus \mathbb{Z}\alpha_{m+n+1}$  be its root lattice. Here  $\alpha_0, \alpha_{m+1}$  are the odd simple roots. The standard invariant form is given by  $(\alpha_i, \alpha_j) = d_i a_{ij}$ , where  $(d_0, d_1, \dots, d_{m+n+1}) = (1, \underbrace{1, \dots, 1}_m, \underbrace{-1, \dots, -1}_{n+1})$ .

We first recall the loop algebra presentation of the 2-toroidal Lie superalgebras.

**THEOREM 2.1.** [12] *The toroidal special linear superalgebra  $T(\mathfrak{g})$  is isomorphic to the Lie superalgebra  $\mathfrak{T}(A(m, n))$  generated by*

$$\{\mathcal{K}, \alpha_i(k), x_i^\pm(k) | 0 \leq i \leq m+n+1, k \in \mathbb{Z}\}$$

with parities given as :  $(0 \leq i \leq m+n+1, k \in \mathbb{Z})$

$$p(\mathcal{K}) = p(\alpha_i(k)) = \bar{0}, \quad p(x_i^\pm(k)) = p(\alpha_i).$$

The defining relations of superbrackets are given by:

- 1)  $[\mathcal{K}, \alpha_i(k)] = [\mathcal{K}, x_i^\pm(k)] = 0;$
- 2)  $[\alpha_i(k), \alpha_j(l)] = k(\alpha_i|\alpha_j)\delta_{k,-l}\mathcal{K};$
- 3)  $[\alpha_i(k), x_j^\pm(l)] = \pm(\alpha_i|\alpha_j)x_j^\pm(k+l);$
- 4)  $[x_i^+(k), x_j^-(l)] = 0, \text{ if } i \neq j;$   
 $[x_i^+(k), x_i^-(l)] = -\{\alpha_i(k+l) + k\delta_{k,-l}\mathcal{K}\}, \text{ if } (\alpha_i|\alpha_i) = 0;$   
 $[x_i^+(k), x_i^-(l)] = -\frac{2}{(\alpha_i|\alpha_i)}\{\alpha_i(k+l) + k\delta_{k,-l}\mathcal{K}\}, \text{ if } (\alpha_i|\alpha_i) \neq 0;$
- 5)  $[x_i^\pm(k), x_i^\pm(l)] = 0;$   
 $[x_i^\pm(k), x_j^\pm(l)] = 0, \text{ if } a_{ii} = a_{ij} = 0, i \neq j;$   
 $[x_i^\pm(k), [x_i^\pm(k), x_j^\pm(l)]] = 0, \text{ if } a_{ii} = 0, a_{ij} \neq 0, i \neq j;$   
 $\underbrace{[x_i^\pm(k), \dots, [x_i^\pm(k), x_j^\pm(l)] \dots]}_{1-a_{ij}} = 0, \text{ if } a_{ii} \neq 0, i \neq j.$

We define formal power series with coefficients from  $\mathfrak{T}(A(m, n))$ :

$$\alpha_i(z) = \sum_{k \in \mathbb{Z}} \alpha_i(k) z^{-k-1}, \quad x_i^\pm(z) = \sum_{k \in \mathbb{Z}} x_i^\pm(k) z^{-k-1},$$

then the defining relations of  $\mathfrak{T}(A(m, n))$  can be rewritten in terms of formal series as follows.

PROPOSITION 2.2. The relations of  $\mathfrak{T}(A(m, n))$  can be written as follows.

- 1')  $[\mathcal{K}, \alpha_i(z)] = [\mathcal{K}, x_i^\pm(z)] = 0;$
- 2')  $[\alpha_i(z), \alpha_j(w)] = (\alpha_i|\alpha_j)\partial_w\delta(z-w)\mathcal{K};$
- 3')  $[\alpha_i(z), x_j^\pm(w)] = \pm(\alpha_i|\alpha_j)x_j^\pm(w)\delta(z-w);$
- 4')  $[x_i^+(z), x_j^-(w)] = 0, \text{ if } i \neq j;$   
 $[x_i^+(z), x_i^-(w)] = -\{(\alpha_i(w)\delta(z-w) + \partial_w\delta(z-w)\mathcal{K}\}, \text{ if } (\alpha_i|\alpha_i) = 0$   
 $[x_i^+(z), x_i^-(w)] = -\frac{2}{(\alpha_i|\alpha_i)}\{(\alpha_i(w)\delta(z-w) + \partial_w\delta(z-w)\mathcal{K}\}, \text{ if } (\alpha_i|\alpha_i) \neq 0$
- 5')  $[x_i^\pm(z), x_i^\pm(w)] = 0;$   
 $[x_i^\pm(z), x_j^\pm(w)] = 0, \text{ if } a_{ii} = a_{ij} = 0, i \neq j;$   
 $[x_i^\pm(z_1), [x_i^\pm(z_2), x_j^\pm(w)]] = 0, \text{ if } a_{ii} = 0, a_{ij} \neq 0, i \neq j;$

$$[x_i^\pm(z_1), \dots, [x_i^\pm(z_{1-a_{ij}}), x_j^\pm(w)] \dots] = 0, \text{ if } a_{ii} \neq 0, i \neq j.$$

Here we have used the formal delta function

$$\delta(z-w) = \sum_{n \in \mathbb{Z}} z^{-n-1} w^n.$$

Its derivatives are given by the power series expansions [13]:

$$\partial_w^{(j)} \delta(z-w) = i_{z,w} \frac{1}{(z-w)^{j+1}} - i_{w,z} \frac{1}{(-w+z)^{j+1}}.$$

where  $\partial_w^{(j)} = \partial_w^j / j!$  and  $i_{z,w}$  means power series expansion in the domain  $|z| > |w|$ . By convention if we write a rational function in the variable  $z-w$  it is usually assumed that the power series is expanded in the region  $|z| > |w|$ . Finally the equation  $f(z,w)\delta(z-w) = f(z,z)\delta(z-w)$  holds when both sides are meaningful.

### 3. Vertex representation of $\mathfrak{T}(A(m,n))$

In this section we will give a representation of the Lie superalgebra  $\mathfrak{T}(A(m,n))$  using both vertex operators and bosonic fields .

Let  $\varepsilon_i$  ( $0 \leq i \leq n+m+3$ ) be an orthonormal basis of the vector space  $\mathbb{C}^{n+m+4}$  and denote by  $\delta_i = \sqrt{-1}\varepsilon_{m+1+i}$  ( $1 \leq i \leq n+2$ ), then the distinguished simple root systems, positive root systems and longest distinguished root of the Lie superalgebra of type  $A(m,n)$  can be represented in terms of vectors  $\varepsilon_i$ 's and  $\delta_i$  's as follows:

$$\begin{aligned} \Pi &= \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \dots, \alpha_m = \varepsilon_m - \varepsilon_{m+1}, \alpha_{m+1} = \varepsilon_{m+1} - \delta_1, \\ &\quad \alpha_{m+2} = \delta_1 - \delta_2, \dots, \alpha_{n+m+1} = \delta_n - \delta_{n+1}\}; \\ \Delta_+ &= \{\varepsilon_i - \varepsilon_j, \delta_k - \delta_l | 1 \leq i < j \leq n+1, 1 \leq k < l \leq m+1\} \\ &\quad \cup \{\delta_k - \varepsilon_i | 1 \leq i \leq n+1, 1 \leq k \leq m+1\}; \\ \theta &= \alpha_1 + \dots + \alpha_{m+n+1} = \varepsilon_1 - \delta_{n+1}. \end{aligned}$$

**3.1. Vertex operators.** Let  $\Gamma = \mathbb{Z}\varepsilon_1 \oplus \dots \oplus \mathbb{Z}\varepsilon_{m+1}$  and  $\mathfrak{h} = \Gamma \otimes_{\mathbb{Z}} \mathbb{C}$ . We view  $\mathfrak{h}$  as an abelian Lie algebra and consider the central extension of its affinization  $\widehat{\mathfrak{h}}$ , i.e.

$$\widehat{\mathfrak{h}} = \bigoplus_{n \neq 0} \mathbb{C}\mathfrak{h} \otimes t^n \oplus \mathbb{C}K$$

with the following commutation relations:

$$[\alpha(k), \beta(l)] = k(\alpha, \beta)\delta_{k,-l}K, \quad [\widehat{\mathfrak{h}}, K] = 0$$

where  $\alpha(k) = \alpha \otimes t^k$  and  $\alpha, \beta \in \Gamma; k, l \in \mathbb{Z}$ . This is an infinite dimensional Heisenberg algebra.

For  $i = 0, 1$ , we let  $\Gamma_{\bar{i}} = \{\alpha \in \Gamma | (\alpha, \alpha) \in 2\mathbb{Z} + i\}$ , then  $\Gamma = \Gamma_{\bar{0}} \oplus \Gamma_{\bar{1}}$ . Let  $F : \Gamma \times \Gamma \rightarrow \{\pm 1\}$  be the bimultiplicative map determined by

$$F(\varepsilon_i, \varepsilon_j) = \begin{cases} 1, & \text{if } i \leq j; \\ -1, & \text{if } i > j. \end{cases}$$

Then the map satisfies the following properties:

- 1)  $F(0, \alpha) = F(\alpha, 0) = 1, \quad \forall \alpha \in \Gamma;$
- 2)  $F(\alpha, \beta)F(\alpha, \beta + \gamma) = F(\beta, \gamma)F(\alpha, \beta + \gamma), \quad \forall \alpha, \beta, \gamma \in \Gamma;$
- 3)  $F(\alpha, \beta)F(\beta, \alpha)^{-1} = (-1)^{(\alpha, \beta) + ij}, \quad \forall \alpha \in \Gamma_{\bar{i}}, \beta \in \Gamma_{\bar{j}}.$

Let  $\mathbb{C}[\Gamma]$  be the vector space spanned by the basis  $\{e^\gamma | \gamma \in \Gamma\}$  over  $\mathbb{C}$ . We define a twisted group algebra structure on  $\mathbb{C}[\Gamma]$  as follows:

$$e^\alpha e^\beta = F(\alpha, \beta) e^{\alpha + \beta}.$$

We form the tensor space

$$V[\Gamma] = \mathbb{C}[\Gamma] \bigotimes S(\bigoplus_{j < 0} (\mathfrak{h} \otimes t^j)),$$

and define the action of  $\widehat{\mathfrak{h}}$  as follows:  $K$  acts as the identity operator,  $\alpha(-k)$  ( $k > 0$ ) acts as multiplication by  $\alpha \otimes t^k$  for  $\alpha \in \Gamma$ , and  $\alpha(k)$  ( $k > 0$ ) acts as the derivation of  $V[\Gamma]$  defined by the formula

$$(3.1a) \quad \alpha(k)(v \otimes e^\beta) = k(\alpha, \beta)(v \otimes e^\beta),$$

$$(3.1b) \quad \begin{aligned} \alpha(k)(e^\beta \otimes \gamma \otimes t^{-l}) \\ = \delta_{k,0}(\alpha, \beta)(e^\beta \otimes \gamma \otimes t^{-l}) + k\delta_{k,l}(\alpha, \gamma)e^\beta \otimes \gamma \otimes t^{-k}. \end{aligned}$$

The space  $V[\Gamma]$  has a natural  $\mathbb{Z}_2$ -gradation:  $V[\Gamma] = V[\Gamma]_{\bar{0}} \oplus V[\Gamma]_{\bar{1}}$ , where  $V[\Gamma]_{\bar{0}}$  (resp.  $V[\Gamma]_{\bar{1}}$ ) is the vector space spanned by  $e^\alpha \otimes \beta \otimes t^{-j}$  with  $\alpha, \beta \in \Gamma; j \in \mathbb{Z}_+$  such that  $(\alpha, \alpha) \in 2\mathbb{Z}$  (resp.  $(\alpha, \alpha) \in 2\mathbb{Z} + 1$ ).

For  $\alpha \in \Gamma$ , we define the vertex operator  $Y(\alpha, z)$  as follows:

$$Y(\alpha, z) = e^\alpha z^{\alpha(0)} \exp(-\sum_{j < 0} \frac{\alpha(j)}{j} z^{-j}), \exp(-\sum_{j > 0} \frac{\alpha(j)}{j} z^{-j})$$

where the operator  $z^{\alpha(0)}$  is given by:

$$z^{\alpha(0)}(e^\beta \otimes \gamma \otimes t^{-j}) = z^{(\alpha, \beta)}(e^\beta \otimes \gamma \otimes t^{-j})$$

for  $\beta, \gamma \in \Gamma; j \in \mathbb{Z}_+$  and denote by

$$X(\alpha, z) = \begin{cases} z^{\frac{(\alpha, \alpha)}{2}} Y(\alpha, z), & \text{if } \alpha \in \Gamma_{\overline{0}}; \\ Y(\alpha, z), & \text{if } \alpha \in \Gamma_{\overline{1}}. \end{cases}$$

We expand  $X(\alpha, z)$  in  $z$

$$X(\alpha, z) = \sum_{j \in \mathbb{Z}} X(\alpha, j) z^{-j-1},$$

where the components  $X(\alpha, j)$  are well-defined local operators. Similarly for  $\alpha \in \Gamma$ , we define

$$\alpha(z) = \sum_{k \in \mathbb{Z}} \alpha(k) z^{-k-1}.$$

LEMMA 3.1. *For  $\alpha \in \Gamma_{\overline{1}}, \beta \in \Gamma_{\overline{1}}$ , one has that*

- 1)  $[Y(\alpha, z), Y(\beta, w)] = 0, \quad \text{if } (\alpha, \beta) \geq 0;$
- 2)  $[Y(\alpha, z), Y(\beta, w)] = F(\alpha, \beta) Y(\alpha + \beta, z) \delta(z - w), \quad \text{if } (\alpha, \beta) = -1;$
- 3)  $[\alpha(z), Y(\beta, w)] = (\alpha, \beta) Y(\beta, z) \delta(z - w).$

PROOF. The first and second part have been proved in [1]. For the third part we refer to [21].  $\square$

COROLLARY 3.2.

- 1)  $[X(\varepsilon_i, z), X(\varepsilon_j - \varepsilon_k, w)] = \delta_{ik} F(\varepsilon_i, \varepsilon_j - \varepsilon_k) X(\varepsilon_j, w) \delta(z - w), \quad j \neq k;$
- 2)  $[X(\varepsilon_i, z), X(-\varepsilon_j, w)] = \delta_{ij} F(\varepsilon_i, -\varepsilon_j) \partial_w \delta(z - w)$
- 3)  $[\alpha(z), X(\beta, w)] = (\alpha, \beta) X(\beta, z) \delta(z - w), \quad \alpha, \beta \in \Gamma;$

PROOF. The corollary is direct result of Lemma 3.1.  $\square$

**3.2. Bosonic fields.** We introduce  $\bar{c} = \varepsilon_0 + \delta_{n+2}$  and define  $\beta = \delta_{n+1} + \bar{c}$ , then  $\alpha_0 = \beta - \varepsilon_1$ . Note that  $(\beta|\beta) = -1, (\beta|\delta_i) = -\delta_{n+1,i}$ . Let  $\mathcal{P}$  be the vector spaces spanned by the set  $\{\bar{c}, \delta_i | 1 \leq i \leq n+1\}$  and  $\mathcal{P}^*$  be its dual space. Let  $\mathcal{C} = \mathcal{P} \oplus \mathcal{P}^*$  and define the bilinear form on it as follows: for  $a, b \in \mathcal{P}$

$$\langle b^*, a \rangle = -\langle a, b^* \rangle = (a, b); \langle b, a \rangle = \langle a^*, b^* \rangle = 0,$$

Let  $\mathcal{A}(\mathbb{Z}^{2n+2})$  be the Weyl algebra generated by  $\{u(k) | u \in \mathcal{C}, k \in \mathbb{Z}\}$  with the defining relations

$$u(k)v(l) - u(k)v(l) = \langle u, v \rangle \delta_{k, -l}$$

for  $u, v \in \mathcal{C}$  and  $k, l \in \mathbb{Z}$ .

The representation space of the algebras  $\mathcal{A}(\mathbb{Z}^{n+1})$  is defined to be the following vector space:

$$\mathfrak{F} = \bigotimes_{a_i} \left( \bigotimes_{k \in \mathbb{Z}_+} \mathbb{C}[a_i(-k)] \bigotimes_{k \in \mathbb{Z}_+} \mathbb{C}[a_i^*(-k)] \right)$$

where  $a_i$  runs through any basis in  $\mathcal{P}$ , consisting of, say  $\bar{c}$  and  $\delta_k$ 's. The algebra  $\mathcal{A}(\mathbb{Z}^{2n+2})$  acts on the space by the usual action:  $a(-k)$  acts as creation operators and  $a(k)$  as annihilation operators.

For  $u \in \mathcal{C}$ , we define the formal power series with coefficients from the associative algebra  $\mathcal{A}(\mathbb{Z}^{2n+2})$ :

$$u(z) = \sum_{k \in \mathbb{Z}} u(k) z^{-k-1}.$$

It is a bosonic field acting on the Fock space  $\mathfrak{F}$ .

In the following, we will give a representation of  $\mathfrak{T}(A(m, n))$  on a quotient  $\mathfrak{V}$  of the tensor space  $V[\Gamma] \otimes \mathfrak{F}$ :

$$\mathfrak{V} = V[\Gamma] \otimes \mathfrak{F} / \left( \sum_k : X(\pm \epsilon_1, -n + k) \bar{c}(n) : \right).$$

Therefore the relation  $: X(\pm \epsilon_1, z) \bar{c}(z) := 0$  holds on  $\mathfrak{V}$ . Note that there is a natural homomorphism from  $\mathfrak{V}$  onto  $V[\bar{\Gamma}] \otimes \mathfrak{F}$ , where  $\bar{\Gamma} = \Gamma / (\epsilon_0 + \delta_{n+2})$ . For simplicity we will use the same symbol to denote the coset elements in  $\mathfrak{V}$ . Observe that there is a  $\mathbb{Z}_2$ -gradation on this space with the parity given by  $p(e^\alpha \otimes x \otimes y) = p(\alpha)$  for  $\alpha \in \Gamma$ ,  $x \in S(\bigoplus_{j < 0} (\mathfrak{h} \otimes t^j))$ ,  $y \in \mathfrak{F}$ . The vertex operators  $X(\alpha, z)$ ,  $\alpha(z)$  acts on the first component and the bosonic fields  $u(z)$  acts on the second component. It follows that

$$p(X(\alpha, z)) = p(\alpha), \quad p(\alpha(z)) = p(u(z)) = \bar{0}.$$

For any two fields  $a(z), b(w)$  with fixed parity, we define the normal ordered product by:

$$\begin{aligned} : a(z) b(w) : &= a(z)_+ b(w) - (-1)^{p(a)p(b)} b(w) a(z)_- \\ &= (-1)^{p(a)p(b)} : b(w) a(z) : \end{aligned}$$

where  $a_\pm(z)$  is defined as usual. Based on the normal ordering of two fields, one can define inductively the normal ordering of more than two fields “from right to left”.

The following facts are well-known in literature, see for example [6, 11].



PROPOSITION 3.3. *One has that*

- 1)  $[\alpha(z), \beta(w)] = (\alpha, \beta) \partial_w \delta(z - w), \quad \alpha, \beta \in \Gamma$
- 2)  $[X(\varepsilon_i - \varepsilon_j, z), X(\varepsilon_j - \varepsilon_i, w)]$   
 $= F(\varepsilon_i - \varepsilon_j, \varepsilon_j - \varepsilon_i)((\varepsilon_i - \varepsilon_j)(z) \delta(z - w) + \partial_w \delta(z - w)),$
- 3)  $:X(\varepsilon_i, z)X(-\varepsilon_j, z): = F(\varepsilon_i, -\varepsilon_j)X(\varepsilon_i - \varepsilon_j, z), \quad i \neq j$
- 4)  $:X(-\varepsilon_j, z)X(\varepsilon_i, z): = F(-\varepsilon_j, -\varepsilon_i)X(\varepsilon_i - \varepsilon_j, z), \quad i \neq j$
- 5)  $:X(\varepsilon_i, z)X(-\varepsilon_i, z): = \varepsilon_i(z).$

Furthermore, we define the contraction of two fields  $a(z), b(w)$  by

$$\underbrace{a(z)b(w)} = a(z)b(w) - :a(z)b(w):.$$

PROPOSITION 3.4. [13] *Suppose fields  $a(z), b(w)$  satisfy the following equality:*

$$[a(z), b(w)] = \sum_{j=0}^{N-1} c^j(w) \partial_w^{(j)} \delta(z - w),$$

where  $N$  is a positive integer and  $c^j(w)$  are formal distributions in the indeterminate  $z$  with value in some algebra related, then we have that

$$\underbrace{a(z)b(w)} = \sum_{j=0}^{N-1} c^j(w) \frac{1}{(z - w)^{j+1}}.$$

The following well-known Wick's theorem is useful for calculating the operator product expansions (OPE) of normally ordered products of free fields.

THEOREM 3.5. [13] *Let  $A^1, A^2, \dots, A^M$  and  $B^1, B^2, \dots, B^N$  be two collections of fields with definite parity. Suppose these fields satisfy the following properties:*

- 1)  $[\underbrace{A^i B^j}, Z^k] = 0, \text{ for all } i, j, k \text{ and } Z = A \text{ or } B;$
- 2)  $[A_{\pm}^i, B_{\pm}^j] = 0, \text{ for all } i, j.$

then we have that

$$\begin{aligned} & :A^1 \dots A^M :: B^1 \dots B^N : \\ &= \sum_{s=0}^m \sum_{\substack{i_1 < \dots < i_s \\ j_1 \neq \dots \neq j_s}} \pm \left( \underbrace{A^{i_1} B^{j_1}} \dots \underbrace{A^{i_s} B^{j_s}} : A^1 \dots A^M B^1 \dots B^N :_{(i_1, \dots, i_s, j_1, \dots, j_s)} \right) \end{aligned}$$

where  $m = \min\{M, N\}$  and the subscript  $(i_1, \dots, i_s, j_1, \dots, j_s)$  means the fields  $A^{i_1}, \dots, A^{i_s}, B^{j_1}, \dots, B^{j_s}$  are removed and the sign  $\pm$  is obtained by the rule: each permutation of the adjacent odd fields changes the sign.

Now we state the main result in this work.

**THEOREM 3.6.** *The following map defines a level one representation on the space  $\mathfrak{V}$ :*

$$x_i^+(z) \mapsto \begin{cases} \sqrt{-1} : X(-\varepsilon_1, z)\beta(z) :, & i = 0; \\ X(\varepsilon_i - \varepsilon_{i+1}, z), & 1 \leq i \leq m; \\ : X(\varepsilon_{m+1}, z)\delta_1^*(z) :, & i = m+1 \\ \sqrt{-1} : \delta_{i-m-1}(z)\delta_{i-m}^*(z) :, & m+2 \leq i \leq m+n+1. \end{cases}$$

$$x_i^-(z) \mapsto \begin{cases} \sqrt{-1} : X(\varepsilon_1, z)\beta^*(z) :, & i = 0; \\ X(\varepsilon_{i+1} - \varepsilon_i, z), & 1 \leq i \leq m; \\ : X(-\varepsilon_{m+1}, z)\delta_1(z) :, & i = m+1 \\ \sqrt{-1} : \delta_{i-m-1}(z)\delta_{i-m}^*(z) :, & m+2 \leq i \leq m+n+1. \end{cases}$$

$$\alpha_i(z) \mapsto \begin{cases} : \beta(z)\beta^*(z) : - \varepsilon_1(z), & i = 0; \\ (\varepsilon_i - \varepsilon_{i+1})(z), & 1 \leq i \leq m; \\ \varepsilon_{m+1}(z) - : \delta_1(z)\delta_1^*(z) :, & i = m+1 \\ : \delta_{i-m-1}(z)\delta_{i-m-1}^*(z) : - : \delta_{i-m}(z)\delta_{i-m}^*(z) :, & m+2 \leq i \leq m+n+1. \end{cases}$$

**PROOF.** To prove the theorem, one needs to check that all the field operators on the right side of above map satisfy relations 1') — 5') listed in Proposition 2.2.

First of all, we check 4') and 3') with the help of Wick's theorem.

$$\begin{aligned} [x_0^+(z), x_0^-(w)] &= - \left( : \beta(z)\beta^*(z) : + : X(-\varepsilon_1, z)X(\varepsilon_1, z) : \right) \delta(z-w) - \partial_w \delta(z-w) \\ &= -(\alpha_0(z)\delta(z-w) + \partial_w \delta(z-w) \cdot 1), \end{aligned}$$

where we have used the fact  $: X(-\varepsilon_1, z)X(\varepsilon_1, z) := -\varepsilon_1(z)$  and

$$[\alpha_0(z), x_0^\pm(w)] = 0 = \pm(\alpha_0, \alpha_0)x_0^\pm(w)\delta(z-w).$$

For  $1 \leq i \leq m$ , we have by Proposition 3.3 that

$$\begin{aligned} [x_i^+(z), x_i^-(w)] &= -((\varepsilon_i - \varepsilon_{i+1})(z)\delta(z-w) + \partial_w \delta(z-w)) \\ &= -\frac{2}{(\alpha_i, \alpha_i)}(\alpha_i(z)\delta(z-w) + \partial_w \delta(z-w) \cdot 1). \end{aligned}$$

It follows from Corollary 3.2 that

$$\begin{aligned}
 [\alpha_i(z), x_i^\pm(w)] &= \pm(\alpha_i, \alpha_i)x_i^\pm(w)\delta(z-w), \\
 [x_{m+1}^+(z), x_{m+1}^-(w)] &= (\delta_1(z)\delta_1^*(z) : - : X(\varepsilon_{m+1}, z)X(-\varepsilon_{m+1}, z) : )\delta(z-w) - \partial_w\delta(z-w) \\
 &= -(\alpha_{m+1}(z)\delta(z-w) + \partial_w\delta(z-w) \cdot 1),
 \end{aligned}$$

and

$$[\alpha_{m+1}(z), x_{m+1}^\pm(w)] = 0 = \pm(\alpha_{m+1}, \alpha_{m+1})x_{m+1}^\pm(w)\delta(z-w).$$

For  $m+2 \leq i \leq m+n+1$ , we have that

$$\begin{aligned}
 [x_i^+(z), x_i^-(w)] &= (\delta_{i-m-1}(z)\delta_{i-m-1}^*(z) : - : \delta_{i-m}(z)\delta_{i-m}^*(z) : )\delta(z-w) + \partial_w\delta(z-w) \\
 &= -\frac{2}{(\alpha_i, \alpha_i)}(\alpha_i(z)\delta(z-w) + \partial_w\delta(z-w) \cdot 1)
 \end{aligned}$$

and

$$\begin{aligned}
 [\alpha_i(z), x_i^+(w)] &= -2\sqrt{-1} : \delta_{i-m-1}(z)\delta_{i-m}^*(z) : \delta(z-w) \\
 &= (\alpha_i, \alpha_i)x_i^+(w)\delta(z-w). \\
 [\alpha_i(z), x_i^-(w)] &= -(\alpha_i, \alpha_i)x_i^-(w)\delta(z-w).
 \end{aligned}$$

For all  $i \neq j$ , we have  $[x_i^+(z), x_j^-(w)] = 0$  and for any unconnected vertices

$$[\alpha_i(z), x_j^\pm(w)] = 0 = \pm(\alpha_i, \alpha_j)x_j^\pm(w)\delta(z-w)$$

All the rest can be checked by straightforward calculation, for examples

$$\begin{aligned}
 [\alpha_0(z), x_1^+(w)] &= -X(\varepsilon_1 - \varepsilon_2, z)\delta(z-w) \\
 &= (\alpha_0, \alpha_1)x_1^+(w)\delta(z-w), \\
 [\alpha_{m+1}(z), x_{m+2}^+(w)] &= : \delta_1(w)\delta_2^*(w) : \delta(z-w) \\
 &= (\alpha_{m+1}, \alpha_{m+2})x_{m+2}^+(w)\delta(z-w), \\
 [\alpha_{m+n+1}(z), x_{m+n}^+(w)] &= \sqrt{-1} : \delta_{n-1}(w)\delta_n^*(w) : \delta(z-w) \\
 &= (\alpha_{m+n+1}, \alpha_{m+n})x_{m+n}^+(w)\delta(z-w).
 \end{aligned}$$

For the extremal vertices one also has that

$$\begin{aligned}
 [\alpha_{m+n+1}(z), x_0^+(w)] &= \sqrt{-1} : X(-\varepsilon_1, w)\delta_{n+1}(w) : \delta(z-w) \\
 &= \sqrt{-1} : X(-\varepsilon_1, w)\beta(w) : \delta(z-w) \\
 &= (\alpha_{m+n+1}, \alpha_0)x_0^+(w)\delta(z-w),
 \end{aligned}$$

where we have used the fact that  $:X(-\varepsilon_1, w)\overline{c}(w) := 0$  and others can be proved similarly.

Secondly, we can check 2') case by case by using Proposition 3.3 1) and we include the following examples

$$\begin{aligned} [\alpha_0(z), \alpha_0(w)] &= 0 = (\alpha_0, \alpha_0) \partial_w \delta(z - w) \cdot 1 \\ [\alpha_0(z), \alpha_1(w)] &= -\partial_w \delta(z - w) = (\alpha_0, \alpha_1) \partial_w \delta(z - w) \cdot 1 \\ [\alpha_0(z), \alpha_{m+n+1}(w)] &= \partial_w \delta(z - w) = (\alpha_0, \alpha_{m+1}) \partial_w \delta(z - w) \cdot 1 \end{aligned}$$

Finally, we proceed to check the Serre relations. It is easy to verify that  $[x_i^\pm(z), x_i^\pm(w)] = 0$  for  $0 \leq i \leq m + n + 1$  and  $[x_i^\pm(z), x_j^\pm(w)] = 0$  for  $i \neq j, a_{ij} = 0$ . The rest can be checked directly:

$$\begin{aligned} &[x_0^+(z_1), [x_0^+(z_2), x_1^+(w)]] \\ &= -[:X(-\varepsilon_1, z_1)\beta(z_1) :, [:X(-\varepsilon_1, z_2)\beta(z_2) :, X(\varepsilon_1 - \varepsilon_2, w)]] \\ &= -[:X(-\varepsilon_1, z_1)\beta(z_1) :, X(-\varepsilon_2, w)]\delta(z_2 - w) \\ &= 0, \end{aligned}$$

$$\begin{aligned} &[x_0^+(z_1), [x_0^+(z_2), x_{m+n+1}^+(w)]] \\ &= -\sqrt{-1}[:X(-\varepsilon_1, z_1)\beta(z_1) :, [:X(-\varepsilon_1, z_2)\beta(z_2) :, \delta_n(w)\delta_{n+1}^*(w) :]] \\ &= -\sqrt{-1}[:X(-\varepsilon_1, z_1)\beta(z_1) :, :X(-\varepsilon_1, w)\delta_n(w) :]\delta(z_2 - w) \\ &= 0, \end{aligned}$$

$$\begin{aligned} &[x_{m+1}^+(z_1), [x_{m+1}^+(z_2), x_m^+(w)]] \\ &= [:X(\varepsilon_{m+1}, z_1)\delta_1^*(z_1) :, [:X(\varepsilon_{m+1}, z_2)\delta_1^*(z_2) :, X(\varepsilon_m - \varepsilon_{m+1}, w)]] \\ &= [:X(\varepsilon_{m+1}, z_1)\delta_1^*(z_1) :, :X(\varepsilon_m, w)\delta_1^*(w) :]\delta(z_2 - w) \\ &= 0, \end{aligned}$$

$$\begin{aligned} &[x_{m+1}^+(z_1), [x_{m+1}^+(z_2), x_{m+2}^+(w)]] \\ &= \sqrt{-1}[:X(\varepsilon_{m+1}, z_1)\delta_1^*(z_1) :, [:X(\varepsilon_{m+1}, z_2)\delta_1^*(z_2) :, \delta_1(w)\delta_2^*(w) :]] \\ &= -\sqrt{-1}[:X(\varepsilon_{m+1}, z_1)\delta_1^*(z_1) :, :X(\varepsilon_{m+1}, w)\delta_2^*(w) :]\delta(z_2 - w) \\ &= 0. \end{aligned}$$

The remaining relations follow similarly by Wick's theorem or Corollary 3.2. This completes the proof of the theorem.  $\square$

## References

- [1] S. Eswara Rao, *Representation of toroidal general linear superalgebras*, Commun. Alg. 42 (2013), 2476–2507.
- [2] S. Eswara Rao, R. V. Moody, *Vertex representations for  $N$ -toroidal Lie algebras and a generalization of the Virasoro algebra*, Commun. Math. Phys., 159 (1994), 239–264.
- [3] S. Bhargava, H. Chen, Y. Gao, *A family of representations of the Lie superalgebra  $\widehat{gl}_{1|l-1}(C_q)$* , J. Algebra 386 (2013), 61–76.
- [4] A. J. Feingold, I. B. Frenkel, *Classical affine algebras*, Adv. Math. 56 (1985), 117–172.
- [5] I. B. Frenkel, N. Jing, W. Wang, *Vertex representations via finite groups and the McKay correspondence*, Int. Math. Res. Notices 4 (2000), 195–222.
- [6] I. B. Frenkel, J. Lepowsky, A. Meuraman, *Vertex Operator algebras and the Monster*, Academic Press, Boston, 1988.
- [7] K. Iohara, Y. Koga, *Central extensions of Lie superalgebras*, Commun. Math. Helv. 76 (2001), 110–154.
- [8] C. Jiang, D. Meng, *Vertex representations for the  $\nu + 1$ -toroidal Lie algebra of type  $B_l$* , J. Alg. 246 (2001), 564–593.
- [9] N. Jing, K. C. Misra, *Fermionic realization of toroidal Lie algebras of classical types*, J. Alg. 324 (2010), 183–194.
- [10] N. Jing, K. C. Misra, S. Tan, *Bosonic realizations of higher level toroidal Lie algebras*, Pacific. J. Math. 219 (2005), 285–302.
- [11] N. Jing, K. C. Misra, C. Xu, *Bosonic realization of toroidal Lie algebras of classical types*, Proc. AMS 137 (2009), 3609–3618.
- [12] N. Jing, C. Xu, *Toroidal Lie superalgebras and free field representations*, Contemp. Math., to appear, arXiv:1308.2826.
- [13] V. G. Kac, *Vertex algebras for beginners*, Univ. Lect. Ser. 10, AMS, Providence, 1997.
- [14] V. G. Kac, M. Wakimoto, *Integrable highest weight modules over affine superalgebras and Appell’s function*, Commun. Math. Phys. 215 (2001), 631–682.
- [15] V. G. Kac, W. Wang, *Vertex operator superalgebras and their representations*, Contemp. Math., 175 (1994) 161–191.
- [16] M. Lau, *Representations of multiloop algebras*, Pacific J. Math. 245 (2010), 167–184.
- [17] D. Liu, N. Hu, *Vertex representations for toroidal Lie algebra of type  $G$* , J. Pure Appl. Alg., 198 (2005), 257–279.
- [18] R. V. Moody, S. E. Rao, T. Yokonuma, *Toroidal Lie algebras and vertex representations*, Geom. Ded. 35 (1990), 283–307.
- [19] S. E. Rao, K. Zhao, *On integrable representations for toroidal Lie superalgebras*, Contemp. Math. 343 (2004), 243–261.
- [20] S. Tan, *Vertex operator representations for toroidal Lie algebra of type  $B_l$* , Commun. Alg. 27 (1999), 3593–3618.

- [21] X. Xu, *Introduction to vertex operator superalgebras and their modules*, Kluwer Academic Publishers, Dordrecht, 1998.

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH,  
NC 27695, USA AND SCHOOL OF MATHEMATICAL SCIENCES, SOUTH CHINA UNIVERSITY  
OF TECHNOLOGY, GUANGZHOU, GUANGDONG 510640, CHINA

*E-mail address:* jing@math.ncsu.edu

SCHOOL OF MATHEMATICAL SCIENCES, SOUTH CHINA UNIVERSITY OF TECHNOLOGY,  
GUANGZHOU, GUANGDONG 510640 AND SCHOOL OF MATHEMATICS & INFORMATION,  
WENZHOU UNIVERSITY, WENZHOU, ZHEJIANG 325035, CHINA

*E-mail address:* xuchongbin1977@126.com